# Blocking estimators and inference under the Neyman-Rubin model

Michael J. Higgins* Fredrik Sävje† Jasjeet S. Sekhon‡

6/15/2015

**Abstract**

We derive the variances of estimators for sample average treatment effects under the Neyman-Rubin potential outcomes model for arbitrary blocking assignments and an arbitrary number of treatments.

## 1 Introduction

Going back to Fisher (1926), the canonical experimental design for preventing imbalances in covariates is *blocking*: where one groups similar experimental units together and assigns treatment in fixed proportions within groups and independently across groups. If blocking reduces imbalances in prognostically important covariates, blocking will improve the expected precision of estimates. As a consequence of the higher accuracy, variance estimators which do not take the design into account will generally overestimate the uncertainty.

*Postdoctoral Research Associate, Program for Quantitative and Analytical Political Science, Princeton University. http://www.princeton.edu/~mjh5/

†PhD Candidate, Department of Economics, Uppsala University. http://fredriksavje.com

‡Robson Professor of Political Science and Statistics. UC Berkeley. http://sekhon.berkeley.edu

Neyman (1935) discusses estimation with a block design under his potential outcomes model—which is often called the Neyman-Rubin causal model (NRCM, Splawa-Neyman et al., 1990; Rubin, 1974; Holland, 1986). In more recent work, variance for the matched-pairs design under NRCM has been analyzed by Abadie and Imbens (2008, looking at conditional average treatment effects, CATE, as estimands) and Imai (2008, population average treatment effects, PATE). Imbens (2011) provides an overview, and discusses PATE, CATE, and the sample average treatment effect (SATE). He also notes the impossibility of variance estimation of the conditional average treatment effect (CATE) with a matched-pairs design.

We build upon this literature and discuss estimation of the sample average treatment effect (SATE) under block designs with arbitrarily many treatment categories and an arbitrary blocking of units. Specially, we derive variances for two unbiased estimators of the SATE—the difference-in-means estimator and the Horvitz-Thompson estimator—and give conservative estimators of these variances whenever block sizes are at least twice the number of treatment categories. Unlike most other methods, these are closed-form formulas of the SATE variance under NRCM applicable to a wide range of settings without making additional parametric assumptions.

# 2 Inference of blocking estimators under Neyman-Rubin model

## 2.1 Notation and preliminaries

There are $n$ units, numbered 1 through $n$. There are $r$ treatments, numbered 1 through $r$. Each unit is assigned to exactly one treatment. Each unit $i$ has a vector of block covariates $\mathbf{x_i}$. A distance between block covariates (such as the Mahalanobis distance) can be computed between each pair of distinct covariates.

Suppose the units are partitioned into $b$ blocks (for example, by our algorithm), numbered 1 through $b$, with each block containing at least $t^*$ units, with $t^* \geq r$. Let $n_c$ denote the number of units in block $c$. Assume that the units within each block $c$ are ordered in some way: let $(k, c)$ denote the $k^{\text{th}}$ unit in block $c$. Let $z$ denote the remainder of $n/r$, and let $z_c$ denote the remainder of $n_c/r$.

#### 2.1.1 Balanced complete and block randomization

Treatment assignment is *balanced* if $z$ treatments are replicated $\lfloor n/r \rfloor + 1$ times, and $r - z$ of the treatments are replicated $\lfloor n/r \rfloor$ times. A balanced treatment assignment is *completely randomized* if each of the

$$\binom{r}{z} \prod_{i=0}^{z-1} \binom{n - i(\lfloor n/r \rfloor + 1)}{\lfloor n/r \rfloor + 1} \prod_{i=0}^{r-z-1} \binom{n - z(\lfloor n/r \rfloor + 1) - i\lfloor n/r \rfloor}{\lfloor n/r \rfloor} \tag{1}$$

possible treatment assignments are equally likely. Treatment is *balanced block randomized* if treatment is balanced and completely randomized within each block and treatment is assigned independently across blocks.

Let $T_{kcs}$ denote treatment indicators for each unit $(k, c)$:

$$T_{kcs} = \begin{cases} 1, & \text{unit } (k,c) \text{ receives treatment } s, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Let $\#T_{cs} = \sum_{k=1}^{n_c} T_{kcs}$ denote the number of units in block $c$ that receive treatment $s$, and let $\#T_s = \sum_{c=1}^{b} \#T_{cs}$ denote the number of units in total assigned to $s$. Let $z_c$ denote the remainder of $n_c/r$.

Under balanced complete randomization, $\#T_s$ has distribution

$$\#T_s = \begin{cases} \lfloor n/r \rfloor + 1 & \text{with probability } z/r, \\ \lfloor n/r \rfloor & \text{with probability } (r-z)/r. \end{cases} \tag{3}$$

Under balanced block randomization, $\#T_{cs}$ has distribution

$$\#T_{cs} = \begin{cases} \lfloor n_c/r \rfloor + 1 & \text{with probability } z_c/r, \\ \lfloor n_c/r \rfloor & \text{with probability } (r-z_c)/r. \end{cases} \tag{4}$$

Since $t^* \geq r$, it follows that $\#T_s \geq \#T_{cs} \geq 1$.

#### 2.1.2 Model for response: the Neyman-Rubin Causal Model

We assume responses follow the Neyman-Rubin Causal Model (NRCM) (Splawa-Neyman et al., 1990; Rubin, 1974; Holland, 1986). Let $y_{kcs}$ denote the *potential outcome* of unit $(k, c)$ given treatment $s$—the hypothetical observed value of unit $(k, c)$ had that unit received treatment $s$. Under the NRCM, the potential outcome $y_{kcs}$ is non-random, and the value of this outcome is observed if and only if $(k, c)$ receives treatment $s$; exactly one of $\{y_{kcs}\}_{s=1}^{r}$ is observed. The observed response is:

$$Y_{kc} \equiv y_{kc1}T_{kc1} + y_{kc2}T_{kc2} + \cdots + y_{kcr}T_{kcr}. \tag{5}$$

Inherent in this equation is the *stable-unit treatment value assumption* (SUTVA): the observed $Y_{kc}$ only depends on which treatment is assigned to unit $(k, c)$, and is not affected by the treatment assignment of any other unit $(k', c')$.

### 2.1.3 Common parameters and estimates under the Neyman-Rubin Causal Model

The domain-level mean and variance of potential outcomes for treatment $s$ are:

$$\mu_s \equiv \frac{1}{n}\sum_{c=1}^{b}\sum_{k=1}^{n_c} y_{kcs}, \tag{6}$$

$$\sigma_s^2 \equiv \sum_{c=1}^{b}\sum_{k=1}^{n_c}\frac{(y_{kcs}-\mu_s)^2}{n} = \sum_{c=1}^{b}\sum_{k=1}^{n_c}\frac{y_{kcs}^2}{n} - \left(\sum_{c=1}^{b}\sum_{k=1}^{n_c}\frac{y_{kcs}}{n}\right)^2, \tag{7}$$

and the domain-level covariance between potential outcomes for treatment $s$ and treatment $t$ is:

$$\begin{aligned}\gamma_{st} &\equiv \sum_{c=1}^{b}\sum_{k=1}^{n_c}\frac{(y_{kcs}-\mu_s)(y_{kct}-\mu_t)}{n}\\ &= \sum_{c=1}^{b}\sum_{k=1}^{n_c}\frac{y_{kcs}y_{kct}}{n} - \left(\sum_{c=1}^{b}\sum_{k=1}^{n_c}\frac{y_{kcs}}{n}\right)\left(\sum_{c=1}^{b}\sum_{k=1}^{n_c}\frac{y_{kct}}{n}\right).\end{aligned} \tag{8}$$

Two estimators for $\mu_s$ are the sample mean and the Horvitz-Thompson estimator (Horvitz and Thompson, 1952):

$$\hat{\mu}_{s,\text{samp}} \equiv \sum_{c=1}^{b}\sum_{k=1}^{n_c}\frac{y_{kcs}T_{kcs}}{\#T_s}, \tag{9}$$

$$\hat{\mu}_{s,\text{HT}} \equiv \sum_{c=1}^{b}\sum_{k=1}^{n_c}\frac{y_{kcs}T_{kcs}}{n/r}. \tag{10}$$

Two estimators for $\sigma_s^2$ are the sample variance and the Horvitz-Thompson estimate of the variance:

$$\hat{\sigma}^2_{s,\text{samp}} \equiv \frac{n-1}{n} \sum_{c=1}^{b} \sum_{k=1}^{n_c} \frac{T_{kcs}\left(y_{kcs} - \sum_{c=1}^{b} \sum_{k=1}^{n_c} \frac{y_{kcs}T_{kcs}}{\#T_s}\right)^2}{\#T_s - 1}. \tag{11}$$

$$\hat{\sigma}^2_{s,\text{HT}} \equiv \frac{(n-1)r}{n^2} \sum_{c=1}^{b} \sum_{k=1}^{n_c} y^2_{kcs} T_{kcs} - \frac{(n-1)r^2}{n^2(n-r) + nz(r-z)} \sum_{(k,c)\neq(k',c')} y_{kcs} y_{k'c's} T_{kcs} T_{k'c's} \tag{12}$$

The sample estimators weight observations by the inverse of the number of observations receiving treatment $s$, and the Horvitz-Thompson estimators weight observations by the inverse of the probability of being assigned treatment $s$. Block-level parameters $\mu_{cs}$, $\sigma_{cs}$, and $\gamma_{cst}$, and block-level estimators $\hat{\mu}_{cs,\text{samp}}$, $\hat{\mu}_{cs,\text{HT}}$, $\hat{\sigma}^2_{cs,\text{samp}}$, and $\hat{\sigma}^2_{cs,\text{HT}}$ are defined as above except that sums range over only units in block $c$.

When treatment is balanced and completely randomized, the domain-level estimators satisfy the following properties:

**Lemma 1** *Under balanced and completely randomized treatment assignment, for any treatments $s$ and $t$ with $s \neq t$,*

$$\mathbb{E}(\hat{\mu}_{s,samp}) = \mu_s, \tag{13}$$

$$Var(\hat{\mu}_{s,samp}) = \frac{r-1}{n-1}\sigma_s^2 + \frac{rz(r-z)}{(n-1)(n-z)(n+r-z)}\sigma_s^2, \tag{14}$$

$$cov(\hat{\mu}_{s,samp}, \hat{\mu}_{t,samp}) = \frac{-\gamma_{st}}{n-1}. \tag{15}$$

**Lemma 2** *Under balanced and completely randomized treatment assignment, for any treat-*

*ments $s$ and $t$ with $s \neq t$,*

$$\begin{aligned}
\mathbb{E}(\hat{\mu}_{s,HT}) &= \mu_s, && (16)\\
Var(\hat{\mu}_{s,HT}) &= \frac{r-1}{n-1}\sigma_s^2 + \frac{z(r-z)}{n^3(n-1)} \sum_{(k,c)\neq(k',c')} y_{kcs}y_{k'c's}, && (17)\\
cov(\hat{\mu}_{s,HT}, \hat{\mu}_{t,HT}) &= \frac{-\gamma_{st}}{n-1} - \frac{z(r-z)}{(r-1)n^3(n-1)} \sum_{(k,c)\neq(k',c')} y_{kcs}y_{k'ct}. && (18)
\end{aligned}$$

**Lemma 3** *Under balanced and completely randomized treatment assignment, for any treatment $s$,*

$$\mathbb{E}(\hat{\sigma}^2_{s,samp}) = \sigma_s^2, \quad \mathbb{E}(\hat{\sigma}^2_{s,HT}) = \sigma_s^2. \tag{19}$$

Recall that, in balanced block randomized designs, treatment is balanced and completely randomized within each block. Thus, analogous properties for block-level estimators hold under balanced block randomization. Lemmas 1 and 2 are proven in Appendix A, and Lemma 3 is proven in Appendix B.

Under the NRCM, the covariance $\gamma_{st}$ is not directly estimable; such an would require estimate requires knowledge of potential outcomes under both treatment $s$ and treatment $t$ within a single unit. However, when blocks contain several replications of each treatment, and when potential outcomes satisfy some smoothness conditions with respect to the block covariates, good estimates of the block-level covariances $\gamma_{cst}$ may be obtained. For details, see Abadie and Imbens (2008); Imbens (2011).

## 2.2 Estimating the sample average treatment effect

Given any two treatments $s$ and $t$, we wish to estimate the *sample average treatment effect of $s$ relative to $t$* ($\text{SATE}_{st}$), denoted $\delta_{st}$. The $\text{SATE}_{st}$ is a sum of differences of potential outcomes:

$$\delta_{st} \equiv \frac{1}{n}\sum_{c=1}^{b}\sum_{k=1}^{n_c}(y_{kcs} - y_{kct}) = \sum_{c=1}^{b}\frac{n_c}{n}\sum_{k=1}^{n_c}\left(\mu_{cs} - \mu_{ct}\right). \tag{20}$$

We consider two estimators for $\delta_{st}$: the difference-in-means estimator:

$$\hat{\delta}_{st,\text{diff}} \equiv \sum_{c=1}^{b}\frac{n_c}{n}\sum_{k=1}^{n_c}\left(\frac{y_{kcs}T_{kcs}}{\#T_{cs}} - \frac{y_{kct}T_{kct}}{\#T_{ct}}\right) = \sum_{c=1}^{b}\frac{n_c}{n}\sum_{k=1}^{n_c}\left(\hat{\mu}_{cs,\text{samp}} - \hat{\mu}_{ct,\text{samp}}\right) \tag{21}$$

and the Horvitz-Thompson estimator (Horvitz and Thompson, 1952):

$$\hat{\delta}_{st,\text{HT}} \equiv \sum_{c=1}^{b}\frac{n_c}{n}\sum_{k=1}^{n_c}\left(\frac{y_{kcs}T_{kcs}}{n_c/r} - \frac{y_{kct}T_{kct}}{n_c/r}\right) = \sum_{c=1}^{b}\frac{n_c}{n}\sum_{k=1}^{n_c}\left(\hat{\mu}_{cs,\text{HT}} - \hat{\mu}_{ct,\text{HT}}\right). \tag{22}$$

These estimators are shown to be unbiased under balanced block randomization in Theorems 4 and 5.

Properties of these estimators are most easily seen by analyzing the block-level terms. Consider first the difference-in-means estimator. By linearity of expectations:

$$\mathbb{E}(\hat{\delta}_{st,\text{diff}}) = \sum_{c=1}^{b}\frac{n_c}{n}(\mathbb{E}(\hat{\mu}_{cs,\text{samp}}) - \mathbb{E}(\hat{\mu}_{ct,\text{samp}})). \tag{23}$$

When treatment is balanced block randomized, by independence of treatment assignment across blocks:

$$\text{Var}(\hat{\delta}_{st,\text{diff}}) = \sum_{c=1}^{b}\frac{n_c^2}{n^2}\left[\text{Var}(\hat{\mu}_{cs,\text{samp}}) + \text{Var}(\hat{\mu}_{ct,\text{samp}}) - 2\text{cov}(\hat{\mu}_{cs,\text{samp}}, \hat{\mu}_{ct,\text{samp}})\right]. \tag{24}$$

Linearity of expectations and independence across blocks can also be exploited to obtain similar expressions hold for the Horvitz-Thompson estimator.

From Lemmas 1 and 2, and using (23) and (24), we can show that both the difference-in-means estimator and the Horvitz-Thompson estimator for the $\text{SATE}_{st}$ are unbiased, and we can compute the variance of these estimates.

**Theorem 4** *Under balanced block randomization, for any treatments $s$ and $t$ with $s \neq t$:*

$$\mathbb{E}(\hat{\delta}_{st,\mathit{diff}}) = \delta_{st}, \tag{25}$$

$$\begin{aligned} \mathit{Var}(\hat{\delta}_{st,\mathit{diff}}) &= \sum_{c=1}^{b} \frac{n_c^2}{n^2} \left( \frac{r-1}{n_c-1}(\sigma_{cs}^2 + \sigma_{ct}^2) + 2\frac{\gamma_{cst}}{n_c-1} \right) \\ &\quad + \sum_{c=1}^{b} \frac{n_c^2}{n^2} \left( \frac{r z_c (r-z_c)}{(n_c-1)(n_c-z_c)(n_c+r-z_c)} (\sigma_{cs}^2 + \sigma_{ct}^2) \right). \end{aligned} \tag{26}$$

**Theorem 5** *Under balanced block randomization, for any treatments $s$ and $t$ with $s \neq t$:*

$$\mathbb{E}(\hat{\delta}_{st,HT}) = \delta_{st}, \tag{27}$$

$$\begin{aligned} \mathit{Var}(\hat{\delta}_{st,HT}) &= \sum_{c=1}^{b} \frac{n_c^2}{n^2} \left( \frac{r-1}{n_c-1}(\sigma_{cs}^2 + \sigma_{ct}^2) + 2\frac{\gamma_{cst}}{n_c-1} \right) \\ &\quad + \sum_{c=1}^{b} \frac{z_c(r-z_c)}{n_c^3(r-1)} \sum_{k=1}^{n_c} \sum_{\ell \neq k} \left( \frac{r-1}{n_c-1} \left( y_{kcs} y_{\ell cs} + y_{kct} y_{\ell ct} \right) + 2\frac{y_{kcs} y_{\ell ct}}{n_c-1} \right) \end{aligned} \tag{28}$$

Note that, when $r$ divides each $n_c$, then $\hat{\delta}_{st,\text{diff}} = \hat{\delta}_{st,\text{HT}}$ and

$$\text{Var}(\hat{\delta}_{st,\text{diff}}) = \text{Var}(\hat{\delta}_{st,\text{HT}}) = \sum_{c=1}^{b} \frac{n_c^2}{n^2} \left( \frac{r-1}{n_c-1}(\sigma_{cs}^2 + \sigma_{ct}^2) + 2\frac{\gamma_{cst}}{n_c-1} \right). \tag{29}$$

When $r$ does not divide each $n_c$, simulation results (not presented) seem to suggest that the difference-in-means estimator has a smaller variance, especially when block sizes are small.

## 2.3 Estimating the variance

As discussed in Section 2.1.3, estimation of the variance for both the difference-in-means and Horvitz-Thompson estimators is complicated by $\gamma_{cst}$ terms, which cannot be estimated without making assumptions about the distribution of potential outcomes. We give conservative estimates (in expectation) for these variances by first deriving unbiased estimators for the block-level variances $\text{Var}(\hat{\mu}_{cs,\text{diff}})$ and $\text{Var}(\hat{\mu}_{cs,\text{HT}})$ and bounding the total variance using the Cauchy-Schwarz inequality and the arithmetic mean/geometric mean (AM-GM) inequality (Hardy et al., 1952) on the covariance terms $\gamma$. These conservative variances make no distributional assumptions on the potential outcomes.

### 2.3.1 Block-level variance estimates

The variance for the block-level estimators (as derived in Lemmas 1 and 2) can be estimated unbiasedly several ways. We consider the following variance estimators.

**Lemma 6** *Define:*

$$\widehat{Var}(\hat{\mu}_{cs,samp}) \equiv \left(\frac{r-1}{n_c-1} + \frac{rz_c(r-z_c)}{(n_c-1)(n_c-z_c)(n_c+r-z_c)}\right)\hat{\sigma}^2_{cs,samp}, \tag{30}$$

$$\begin{aligned}\widehat{Var}(\hat{\mu}_{cs,HT}) \equiv\ & \frac{r-1}{n_c-1}\hat{\sigma}^2_{cs,HT} \\ & + \frac{r^2 z_c(r-z_c)}{n_c^3(n_c-r)+n_c^2 z_c(r-z_c)}\sum_{k=1}^{n_c}\sum_{\ell\neq k} y_{kcs}y_{\ell cs}T_{kcs}T_{\ell cs},\end{aligned} \tag{31}$$

*Under balanced block randomization, for any treatment $s$:*

$$\mathbb{E}\left[\widehat{Var}(\hat{\mu}_{cs,samp})\right] = Var(\hat{\mu}_{cs,samp}), \quad \mathbb{E}\left[\widehat{Var}(\hat{\mu}_{cs,HT}))\right] = Var(\hat{\mu}_{cs,HT}). \tag{32}$$

This Lemma is proven in Appendix B.

#### 2.3.2 Conservative variance estimates for SATE$_{st}$ estimators

Define the following variance estimators:

$$\widehat{\text{Var}}(\hat{\delta}_{st,\text{diff}}) \equiv \sum_{c=1}^{b} \frac{n_c^2}{n^2}\left[\left(\frac{r}{n_c-1} + \frac{rz_c(r-z_c)}{(n_c-1)(n_c-z_c)(n_c+r-z_c)}\right)(\hat{\sigma}^2_{cs,\text{samp}} + \hat{\sigma}^2_{ct,\text{samp}})\right], \tag{33}$$

$$\begin{aligned}\widehat{\text{Var}}(\hat{\delta}_{st,\text{HT}}) \equiv \sum_{c=1}^{b} \frac{n_c^2}{n^2}\Bigg[&\frac{r}{n_c-1}(\hat{\sigma}^2_{cs,\text{HT}} + \hat{\sigma}^2_{ct,\text{HT}}) \\ &+ \frac{r^2 z_c(r-z_c)}{n_c^3(n_c-r)+n_c^2 z_c(r-z_c)} \sum_{k=1}^{n_c}\sum_{\ell\neq k}(y_{kcs}y_{\ell cs}T_{kcs}T_{\ell cs} + y_{kct}y_{\ell ct}T_{kct}T_{\ell ct}) \\ &+ \frac{2r^2 z_c(r-z_c)}{n_c^4(r-1)-n_c^2 z_c(r-z_c)} \sum_{k=1}^{n_c}\sum_{\ell\neq k} y_{kcs}y_{\ell ct}T_{kcs}T_{\ell ct}\Bigg].\end{aligned} \tag{34}$$

We now show that these estimators are conservative (in expectation). First, we begin with the following lemma:

**Lemma 7** *Under balanced block randomization, for any treatments $s$ and $t$ with $s \neq t$:*

$$\begin{aligned}&\mathbb{E}\left(\frac{2r^2 z_c(r-z_c)}{n_c^4(r-1)-n_c^2 z(r-z)} \sum_{k=1}^{n_c}\sum_{\ell\neq k} y_{kcs}y_{\ell ct}T_{kcs}T_{\ell ct}\right) \\ &= \frac{2z_c(r-z_c)}{n_c^3(n_c-1)(r-1)} \sum_{k=1}^{n_c}\sum_{\ell\neq k} y_{kcs}y_{\ell ct}.\end{aligned} \tag{35}$$

This lemma is proved in Appendix B.

Also note, by the Cauchy-Schwarz and the AM-GM inequalities respectively, that:

$$\gamma_{cst} \leq \sqrt{\sigma^2_{cs}\sigma^2_{ct}} \leq \frac{\sigma^2_{cs} + \sigma^2_{ct}}{2}. \tag{36}$$

The first two terms are equal if and only if there exists constants $a$ and $b$ such that, for all $k \in \{1, \ldots, n_c\}$, $y_{kcs} = a + by_{kct}$. The last two terms are equal if and only if $\sigma^2_{cs} = \sigma^2_{ct}$. Hence, (36) is satisfied with equality if and only if there exists a constant $a$ such that, for all $k \in \{1, \ldots, n_c\}$, $y_{kcs} = a + y_{kct}$; that is, if and only if treatment shifts the value of the potential outcomes by a constant for all units within block $c$.

**Theorem 8** *Under balanced block randomization, for any treatments $s$ and $t$ with $s \neq t$:*

$$\mathbb{E}(\widehat{Var}(\hat{\delta}_{st,diff})) \geq Var(\hat{\delta}_{st,diff}), \quad \mathbb{E}(\widehat{Var}(\hat{\delta}_{st,HT})) \geq Var(\hat{\delta}_{st,HT}). \tag{37}$$

*with equality if and only if, for each block $c$, there is a constant $a_c$ such that, for all $k \in \{1, \ldots, n_c\}$, $y_{kcs} = a_c + y_{kct}$. $\sigma^2_{cs} = \sigma^2_{ct}$.*

**Proof:** Define:

$$\begin{aligned}
\text{Var}^*_{st,\text{diff}} \;&\equiv\; \sum_{c=1}^{b} \frac{n_c^2}{n^2}\left(\frac{r}{n_c-1}(\sigma^2_{cs}+\sigma^2_{ct})\right) \\
&\quad + \sum_{c=1}^{b} \frac{n_c^2}{n^2}\left(\frac{rz_c(r-z_c)}{(n_c-1)(n_c-z_c)(n_c+r-z_c)}(\sigma^2_{cs}+\sigma^2_{ct})\right) \qquad (38)\\
\text{Var}^*_{st,\text{HT}} \;&\equiv\; \sum_{c=1}^{b} \frac{n_c^2}{n^2}\left(\frac{r}{n_c-1}(\sigma^2_{cs}+\sigma^2_{ct})\right) \\
&\quad + \sum_{c=1}^{b} \frac{z_c(r-z_c)}{n_c^3(r-1)} \sum_{k=1}^{n_c}\sum_{\ell\neq k}\left(\frac{r-1}{n_c-1}\left(y_{kcs}y_{\ell cs}+y_{kct}y_{\ell ct}\right)+2\frac{y_{kcs}y_{\ell ct}}{n_c-1}\right)
\end{aligned} \tag{39}$$

By Theorems 4 and 5, and by equation (36), it follows that:

$$\text{Var}^*_{st,\text{diff}} \geq \text{Var}(\hat{\delta}_{st,\text{diff}}), \quad \text{Var}^*_{st,\text{HT}} \geq \text{Var}(\hat{\delta}_{st,\text{HT}}), \tag{40}$$

with equality if and only if, for each block $c$, there is a constant $a_c$ such that, for all $k \in \{1, \dots, n_c\}$, $y_{kcs} = a_c + y_{kct}$. Moreover, by Lemmas 6 and 7, and by linearity of expectations, we have that:

$$\mathbb{E}(\widehat{\text{Var}}(\hat{\delta}_{st,\text{diff}})) = \text{Var}^*_{st,\text{diff}}, \quad \mathbb{E}(\widehat{\text{Var}}(\hat{\delta}_{st,\text{HT}})) = \text{Var}^*_{st,\text{HT}}. \tag{41}$$

The theorem immediately follows. ■

## 2.4 Comparing block randomization and complete randomization

We now describe conditions under which the variance of $\text{SATE}_{st}$ estimates under balanced block randomization is smaller than those under balanced complete randomization (without blocking). We then show that these conditions are met (in expectation) when the assignment of units into blocks of fixed size is completely randomized. Thus, unless block covariates are worse than random chance at predicting potential outcomes, blocking will only improve precision of $\text{SATE}_{st}$ estimates. These results are a generalization of those found in Imai (2008). To make the mathematics more tractable, we only consider the case where $r$ divides each $n_c$.

When treatment assignment is balanced and completely randomized, the following estimator is always unbiased for the $\text{SATE}_{st}$:

$$\hat{\delta}_{st,\text{cr}} \equiv \frac{1}{n/r} \sum_{c=1}^{b} \sum_{k=1}^{n_c} y_{kcs}T_{kcs} - y_{kct}T_{kct}. \tag{42}$$

When $r$ divides each $n_c$ (and hence, $r$ divides $n$), this estimator is the same as $\hat{\delta}_{st,\text{diff}}$ and $\hat{\delta}_{st,\text{HT}}$. Hence, under these assumptions, this estimator has variance:

$$\begin{aligned}
\text{Var}(\hat{\delta}_{st,\text{cr}}) &= \frac{r-1}{n-1}(\sigma_s^2 + \sigma_t^2) + 2\frac{\gamma_{st}}{n-1} \\
&= \sum_{c=1}^{b} \frac{n_c^2}{\sum_c n_c^2}\left(\frac{r-1}{n-1}(\sigma_s^2 + \sigma_t^2) + 2\frac{\gamma_{st}}{n-1}\right). \qquad (43)
\end{aligned}$$

Proofs of (42) and (43) follow those in Appendix A.

Suppose an experimenter has already partitioned experimental units into blocks and is deciding between completely randomizing treatment and block randomizing treatment. By (29) and (43), when $r$ divides each $n_c$, the variance of $\text{SATE}_{st}$ estimators under block randomization will be as small or smaller than that under complete randomization precisely when

$$\sum_{c=1}^{b} \frac{n_c^2}{\sum n_c^2}\left(\frac{r-1}{n-1}(\sigma_s^2 + \sigma_t^2) + 2\frac{\gamma_{st}}{n-1}\right) - \frac{n_c^2}{n^2}\left(\frac{r-1}{n_c-1}(\sigma_{cs}^2 + \sigma_{ct}^2) + 2\frac{\gamma_{cst}}{n_c-1}\right) \geq 0. \qquad (44)$$

We can write this condition in terms of a comparison between block-level variances and sample-level variances. For all units $(k, c)$, define $y_{kc(s+t)} \equiv y_{kcs} + y_{kct}$. Let $\sigma_{s+t}^2$ and $\sigma_{c(s+t)}^2$ denote the domain-level and block-level variance of these $y_{kc(s+t)}$ as defined in (7). It follows that the variance under block randomization will be as small or smaller than that under complete randomization if and only if

$$\delta_{\text{cr,blk}} \equiv \sum_{c=1}^{b} n_c^2 \left(\frac{(r-2)(\sigma_s^2 + \sigma_t^2) + \sigma_{s+t}^2}{(n-1)\sum_c n_c^2} - \frac{(r-2)(\sigma_{cs}^2 + \sigma_{ct}^2) + \sigma_{c(s+t)}^2}{(n_c-1)n^2}\right) \geq 0 \qquad (45)$$

This formula gives some insight as to what properties of a blocking are helpful in reducing variance. Terms of $\delta_{\text{cr,bl}}$ will be positive (and thus, will favor estimates under block randomization) if and only if

$$\frac{(r-2)(\sigma^2_{cs}+\sigma^2_{ct})+\sigma^2_{c(s+t)}}{(r-2)(\sigma^2_{s}+\sigma^2_{t})+\sigma^2_{s+t}} \leq \frac{(n-1)\sum_c n_c^2}{(n_c-1)n^2} \tag{46}$$

Since the right-hand-side fraction gets smaller as $n_c$ gets larger, it follows that blocking helps most when the block-level variances in the largest-sized blocks is small.

We now show that, when units are randomly assigned to blocks, the precision of estimates of the $\text{SATE}_{st}$ will be the same in expectation under either complete randomization or block randomization. More formally, we say that an assignment of $n$ units into blocks of sizes $\mathbf{n} = (n_1, \ldots, n_b)$ is a *completely randomized blocking with block sizes* $\mathbf{n}$ if each possible blocking with those block sizes is equally likely. Under completely randomized blocking, the block-level variances $\sigma^2_{cs}$, $\sigma^2_{ct}$, and $\sigma^2_{c(s+t)}$ are random variables; sample-level variances $\sigma^2_s$, $\sigma^2_t$, and $\sigma^2_{s+t}$ and block sizes are constants. In Appendix C, we prove the following theorem.

**Theorem 9** *Under completely randomized blocking, when $r$ divides each $n_c$,*

$$\mathbb{E}(\delta_{cr,blk}) = 0. \tag{47}$$

That is, even when units are assigned to blocks randomly, the variance of an estimate of the $\text{SATE}_{st}$ under block randomization be no larger than that under complete randomization. When block covariates predict potential outcomes better than at random, blocking guarantees an increase the precision of $\text{SATE}_{st}$ estimates.

# A Proof of Lemmas 1 and 2

The following proofs use methods found in Cochran (1977) and Lohr (1999). Additionally, the variance calculations for the sample mean follow Miratrix et al. (2013) closely. To help the reader, we refer each unit by a single index.

For any distinct units $i$ and $j$, and distinct treatments $s$ and $t$, the following expectations hold under complete randomization:

$$
\begin{aligned}
\mathbb{E}\left(\frac{T_{is}}{\#T_s}\right) &= \mathbb{E}\left[\mathbb{E}\left(\frac{T_{is}}{\#T_s}\middle|\#T_s\right)\right] \\
&= \mathbb{E}\left(\frac{\frac{\#T_s}{n}}{\#T_s}\right) = \mathbb{E}\left(\frac{1}{n}\right) = \frac{1}{n}, \quad (48) \\
\mathbb{E}\left(\frac{T_{is}}{(\#T_s)^2}\right) &= \mathbb{E}\left[\mathbb{E}\left(\frac{T_{is}}{(\#T_s)^2}\middle|\#T_{is}\right)\right] \\
&= \mathbb{E}\left(\frac{\frac{\#T_s}{n}}{(\#T_s)^2}\right) = \mathbb{E}\left(\frac{1}{n}\frac{1}{\#T_s}\right) = \frac{1}{n}\mathbb{E}\left(\frac{1}{\#T_s}\right), \quad (49) \\
\mathbb{E}\left(\frac{T_{is}T_{js}}{(\#T_s)^2}\right) &= \mathbb{E}\left[\mathbb{E}\left(\frac{T_{is}T_{js}}{(\#T_s)^2}\middle|\#T_{is}\right)\right] \\
&= \mathbb{E}\left(\frac{\frac{\#T_s}{n}\frac{\#T_s-1}{n-1}}{(\#T_s)^2}\right) = \mathbb{E}\left(\frac{(\#T_s)^2-\#T_s}{n(n-1)(\#T_s)^2}\right) \\
&= \frac{1}{n(n-1)}\mathbb{E}\left(1-\frac{1}{\#T_s}\right) \\
&= \frac{1}{n(n-1)} - \frac{1}{n(n-1)}\mathbb{E}\left(\frac{1}{\#T_s}\right), \quad (50) \\
\mathbb{E}\left(\frac{T_{is}T_{jt}}{\#T_s\#T_t}\right) &= \mathbb{E}\left[\mathbb{E}\left(\frac{T_{is}T_{jt}}{\#T_s\#T_t}\middle|\#T_s,\#T_t\right)\right] \\
&= \mathbb{E}\left(\frac{\frac{\#T_s}{n}\frac{\#T_t}{n-1}}{\#T_s\#T_t}\right) = \mathbb{E}\left(\frac{1}{n(n-1)}\right) = \frac{1}{n(n-1)}. \quad (51)
\end{aligned}
$$

We first compute the expectation of the block-level estimator $\hat{\mu}_{s,\text{samp}}$. By 48,

$$
\mathbb{E}(\hat{\mu}_{s,\text{samp}}) = \mathbb{E}\left(\sum_{i=1}^{n}\frac{y_{is}T_{is}}{\#T_s}\right) = \sum_{i=1}^{n} y_{is}\mathbb{E}\left(\frac{T_{is}}{\#T_s}\right) = \sum_{i=1}^{n}\frac{y_{is}}{n} = \mu_s. \quad (52)
$$

We now derive the variance of this estimator. Observe that, by (49) and (50):

$$\begin{aligned}
&\mathbb{E}\left(\hat{\mu}_{s,\text{diff}}^2\right) = \mathbb{E}\left[\left(\sum_{i=1}^{n} \frac{y_{is}T_{is}}{\#T_s}\right)^2\right] \\
= \;& \mathbb{E}\left[\sum_{i=1}^{n} \frac{y_{is}^2 T_{is}^2}{(\#T_s)^2} + \sum_{i=1}^{n}\sum_{j\neq i} \frac{y_{is}y_{js}T_{is}T_{js}}{(\#T_s)^2}\right] = \mathbb{E}\left[\sum_{i=1}^{n} \frac{y_{is}^2 T_{is}}{(\#T_s)^2} + \sum_{i=1}^{n}\sum_{j\neq i} \frac{y_{is}y_{js}T_{is}T_{js}}{(\#T_s)^2}\right] \\
= \;& \sum_{i=1}^{n} y_{is}^2 \mathbb{E}\left(\frac{T_{is}}{(\#T_s)^2}\right) + \sum_{i=1}^{n}\sum_{j\neq i} y_{is}y_{js}\mathbb{E}\left(\frac{T_{is}T_{js}}{(\#T_s)^2}\right) \\
= \;& \frac{1}{n}\mathbb{E}\left(\frac{1}{\#T_s}\right)\sum_{i=1}^{n} y_{is}^2 + \sum_{i=1}^{n}\sum_{j\neq i} y_{is}y_{js}\left(\frac{1}{n(n-1)} - \frac{1}{n(n-1)}\mathbb{E}\left(\frac{1}{\#T_s}\right)\right) \\
= \;& \frac{1}{n}\mathbb{E}\left(\frac{1}{\#T_s}\right)\sum_{i=1}^{n} y_{is}^2 + \left(\left(\sum_{i=1}^{n} y_{is}\right)^2 - \sum_{i=1}^{n} y_{is}^2\right)\left(\frac{1}{n(n-1)} - \frac{1}{n(n-1)}\mathbb{E}\left(\frac{1}{\#T_s}\right)\right) \\
= \;& \frac{1}{n(n-1)}\left(\sum_{i=1}^{n} y_{is}\right)^2 - \frac{1}{n(n-1)}\sum_{i=1}^{n} y_{is}^2 \\
& + \mathbb{E}\left(\frac{1}{\#T_s}\right)\left(\left(\frac{1}{n} + \frac{1}{n(n-1)}\right)\sum_{i=1}^{n} y_{is}^2 - \frac{1}{n(n-1)}\left(\sum_{i=1}^{n} y_{is}\right)^2\right). \qquad (53)
\end{aligned}$$

We can simplify the last term in parentheses.

$$\begin{aligned}
&\left(\frac{1}{n} + \frac{1}{n(n-1)}\right)\sum_{i=1}^{n} y_{is}^2 - \frac{1}{n(n-1)}\left(\sum_{i=1}^{n} y_{is}\right)^2 \\
= \;& \frac{1}{n-1}\sum_{i=1}^{n} y_{is}^2 - \frac{1}{n(n-1)}\left(\sum_{i=1}^{n} y_{is}\right)^2 \\
= \;& \frac{1}{n-1}\sum_{i=1}^{n} y_{is}^2 - \frac{n}{n-1}\left(\sum_{i=1}^{n} \frac{y_{is}}{n}\right)^2 \\
= \;& \frac{n}{n-1}\sum_{i=1}^{n} \frac{y_{is}^2}{n} - \frac{n}{n-1}\left(\sum_{i=1}^{n} \frac{y_{is}}{n}\right)^2 \\
= \;& \frac{n}{n-1}\left(\sum_{i=1}^{n} \frac{y_{is}^2}{n} - \left(\sum_{i=1}^{n} \frac{y_{is}}{n}\right)^2\right) = \frac{n}{n-1}\sigma_s^2. \qquad (54)
\end{aligned}$$

The last equality is obtained by applying (7).

Continuing from (53) and applying (54), we find that:

$$\begin{aligned}
\mathbb{E}\left(\hat{\mu}^2_{s,\text{diff}}\right) &= \frac{1}{n(n-1)}\left(\sum_{i=1}^{n} y_{is}\right)^2 - \frac{1}{n(n-1)}\sum_{i=1}^{n} y_{is}^2 \\
&\quad + \mathbb{E}\left(\frac{1}{\#T_s}\right)\left(\left(\frac{1}{n}+\frac{1}{n(n-1)}\right)\sum_{i=1}^{n} y_{is}^2 - \frac{1}{n(n-1)}\left(\sum_{i=1}^{n} y_{is}\right)^2\right) \\
&= \frac{n}{n-1}\left(\sum_{i=1}^{n}\frac{y_{is}}{n}\right)^2 - \frac{1}{n-1}\sum_{i=1}^{n}\frac{y_{is}^2}{n} + \mathbb{E}\left(\frac{1}{\#T_s}\right)\frac{n}{n-1}\sigma_s^2. \qquad (55)
\end{aligned}$$

Since $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$, it follows from (55) and (7) that:

$$\begin{aligned}
\text{Var}(\hat{\mu}_{s,\text{diff}}) &= \mathbb{E}(\hat{\mu}_s^2) - (\mathbb{E}(\hat{\mu}_s))^2 \\
&= \frac{n}{n-1}\left(\sum_{i=1}^{n}\frac{y_{is}}{n}\right)^2 - \frac{1}{n-1}\sum_{i=1}^{n}\frac{y_{is}^2}{n} + \mathbb{E}\left(\frac{1}{\#T_s}\right)\frac{n}{n-1}\sigma_s^2 - \left(\sum_{i=1}^{n}\frac{y_{is}}{n}\right)^2 \\
&= \frac{1}{n-1}\left(\sum_{i=1}^{n}\frac{y_{is}}{n}\right)^2 - \frac{1}{n-1}\sum_{i=1}^{n}\frac{y_{is}^2}{n} + \mathbb{E}\left(\frac{1}{\#T_s}\right)\frac{n}{n-1}\sigma_s^2 \\
&= \frac{-1}{n-1}\left(\frac{1}{n-1}\sum_{i=1}^{n}\frac{y_{is}^2}{n} - \left(\sum_{i=1}^{n}\frac{y_{is}}{n}\right)^2\right) + \mathbb{E}\left(\frac{1}{\#T_s}\right)\frac{n}{n-1}\sigma_s^2 \\
&= \frac{-1}{n-1}\sigma_s^2 + \mathbb{E}\left(\frac{1}{\#T_s}\right)\frac{n}{n-1}\sigma_s^2 = \frac{n}{n-1}\left(\mathbb{E}\left(\frac{1}{\#T_s}\right) - \frac{1}{n}\right)\sigma_s^2. \qquad (56)
\end{aligned}$$

Note that $\lfloor n/r \rfloor = n/r - z/r$. Thus, under complete randomization:

$$\begin{aligned}
\mathbb{E}\left(\frac{1}{\#T_s}\right) &= \frac{z}{r}\left(\frac{1}{\lfloor n/r \rfloor + 1}\right) + \left(1 - \frac{z}{r}\right)\left(\frac{1}{\lfloor n/r \rfloor}\right) \\
&= \frac{z\lfloor n/r \rfloor}{r(\lfloor n/r \rfloor)(\lfloor n/r \rfloor + 1)} + \frac{(r-z)(\lfloor n/r \rfloor + 1)}{r(\lfloor n/r \rfloor)(\lfloor n/r \rfloor + 1)} \\
&= \frac{z(n/r - z/r) + (r-z)(n/r - z/r + 1)}{r(n/r - z/r)(n/r - z/r + 1)} \\
&= \frac{(1/r)z(n-z) + (1/r)(r-z)(n-z+r)}{(1/r)(n-z)(n+r-z)} \\
&= \frac{z(n-z) + (r-z)(n-z+r)}{(n-z)(n+r-z)} \\
&= \frac{r(n-z) + r^2 - zr}{(n-z)(n+r-z)} = \frac{nr + r^2 - 2rz}{(n-z)(n+r-z)}. \qquad (57)
\end{aligned}$$

It follows that:

$$\begin{aligned}
\mathrm{Var}(\hat{\mu}_{s,\text{diff}}) &= \frac{n}{n-1}\sigma_s^2\left(\mathbb{E}\left(\frac{1}{\#T_s}\right) - \frac{1}{n}\right) \\
&= \frac{n}{n-1}\sigma_s^2\left(\frac{nr + r^2 - 2rz}{(n-z)(n+r-z)} - \frac{1}{n}\right) \\
&= \frac{n}{n-1}\frac{n^2r + nr^2 - 2nrz - (n-z)(n+r-z)}{n(n-z)(n+r-z)}\sigma_s^2 \\
&= \frac{nr(n+r-z) - nrz - (n-z)(n+r-z)}{(n-1)(n-z)(n+r-z)}\sigma_s^2 \\
&= \frac{nr(n+r-z) - rz(n+r-z) - (n-z)(n+r-z) + rz(r-z)}{(n-1)(n-z)(n+r-z)}\sigma_s^2 \\
&= \frac{(nr - rz - n + z)(n+r-z) + rz(r-z)}{(n-1)(n-z)(n+r-z)}\sigma_s^2 \\
&= \frac{(r-1)(n-z)(n+r-z) + rz(r-z)}{(n-1)(n-z)(n+r-z)}\sigma_s^2 \\
&= \left(\frac{r-1}{n-1} + \frac{rz(r-z)}{(n-1)(n-z)(n+r-z)}\right)\sigma_s^2. \qquad (58)
\end{aligned}$$

We now derive covariances of this estimator. Note that:

$$\begin{aligned}
& \mathbb{E}\left(\sum_{i=1}^n \frac{y_{is}T_{is}}{\#T_s}\sum_{i=1}^n \frac{y_{it}T_{it}}{\#T_t}\right) \\
= \; & \mathbb{E}\left(\sum_{i=1}^n \sum_{j\neq i} \frac{y_{is}y_{jt}T_{is}T_{jt}}{\#T_s\#T_t}\right) + \mathbb{E}\left(\sum_{i=1}^n \frac{y_{is}y_{it}T_{is}T_{it}}{\#T_s\#T_t}\right) \\
= \; & \sum_{i=1}^n \sum_{j\neq i} y_{is}y_{jt}\mathbb{E}\left(\frac{T_{is}T_{jt}}{\#T_s\#T_t}\right) + \sum_{i=1}^n y_{is}y_{it}\mathbb{E}\left(\frac{T_{is}T_{it}}{\#T_s\#T_t}\right) \\
= \; & \sum_{i=1}^n \sum_{j\neq i} y_{is}y_{jt}\frac{1}{n(n-1)} + 0 = \sum_{i=1}^n \sum_{j\neq i} \frac{y_{is}y_{jt}}{n(n-1)}. \qquad (59)
\end{aligned}$$

Recall $\text{cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$. It follows that:

$$\begin{aligned}
\text{cov}(\hat{\mu}_{s,\text{diff}}, \hat{\mu}_{t,\text{diff}}) &= \mathbb{E}\left(\sum_{i=1}^n \frac{y_{is}T_{is}}{\#T_{is}}\sum_{i=1}^n \frac{y_{it}T_{it}}{\#T_{it}}\right) - \sum_{i=1}^n \frac{y_{is}}{n}\sum_{i=1}^n \frac{y_{it}}{n} \\
&= \sum_{i=1}^n \sum_{j\neq i} \frac{y_{is}y_{jt}}{n(n-1)} - \sum_{i=1}^n \frac{y_{is}}{n}\sum_{i=1}^n \frac{y_{it}}{n} \\
&= \frac{1}{n(n-1)}\sum_{i=1}^n y_{is}\sum_{i=1}^n y_{it} - \frac{1}{n(n-1)}\sum_{i=1}^n y_{is}y_{it} - \frac{1}{n^2}\sum_{i=1}^n y_{is}\sum_{i=1}^n y_{it} \\
&= \frac{1}{n^2(n-1)}\sum_{i=1}^n y_{is}\sum_{i=1}^n y_{it} - \frac{1}{n(n-1)}\sum_{i=1}^n y_{is}y_{it} \\
&= \frac{-1}{n-1}\left(\sum_{i=1}^n \frac{y_{is}y_{it}}{n} - \sum_{i=1}^n \frac{y_{is}}{n}\sum_{i=1}^n \frac{y_{it}}{n}\right) = \frac{-\gamma_{st}}{n-1}. \qquad (60)
\end{aligned}$$

Our derivation of the variance and covariance of the sample mean show that the variance and covariate expressions derived in Miratrix et al. (2013) are incorrect by a factor of $\frac{n}{n-1}$.

We now turn our attention to the Horvitz-Thompson estimator. For any distinct units $i$ and $j$, and distinct treatments $s$ and $t$, the following expectations hold under complete randomization:

$$
\begin{aligned}
\mathbb{E}\left(\#T_s\right) &= \mathbb{E}\left(\sum_{i=1}^{n} T_{is}\right) = \sum_{i=1}^{n} \mathbb{E}(T_{is}) = \sum_{i=1}^{n} 1/r = n/r, && (61)\\
\mathbb{E}\left((\#T_s)^2\right) &= \frac{z}{r}\left(\lfloor n/r\rfloor + 1\right)^2 + \left(1-\frac{z}{r}\right)\left(\lfloor n/r\rfloor\right)^2\\
&= \frac{z}{r}\left((\lfloor n/r\rfloor)^2 + 2\lfloor n/r\rfloor + 1\right) + \left(1-\frac{z}{r}\right)\left(\lfloor n/r\rfloor\right)^2\\
&= (\lfloor n/r\rfloor)^2 + \frac{2z}{r}\lfloor n/r\rfloor + \frac{z}{r} = (\lfloor n/r\rfloor + z/r)^2 + (z/r - (z/r)^2)\\
&= (n/r)^2 + z/r(1-z/r), && (62)\\
\mathbb{E}\left(\#T_s\#T_t\right) &= \frac{z(z-1)}{r(r-1)}\left(\lfloor n/r\rfloor + 1\right)^2 + \frac{(r-z)(r-z-1)}{r(r-1)}\left(\lfloor n/r\rfloor\right)^2\\
&\quad + \frac{2z(r-z)}{r(r-1)}\left(\lfloor n/r\rfloor + 1\right)\left(\lfloor n/r\rfloor\right)\\
&= \frac{1}{r(r-1)}\left(\begin{array}{l} z(z-1)((\lfloor n/r\rfloor)^2 + 2\lfloor n/r\rfloor + 1)\\ +(r-z)(r-z-1)(\lfloor n/r\rfloor)^2\\ +2z(r-z)((\lfloor n/r\rfloor)^2 + \lfloor n/r\rfloor)\end{array}\right)\\
&= \frac{1}{r(r-1)}\left(\begin{array}{l} (z(z-1) + (r-z)(r-z-1) + 2z(r-z))(\lfloor n/r\rfloor)^2\\ +(2z(z-1) + 2z(r-z))\lfloor n/r\rfloor\\ +z(z-1)\end{array}\right)\\
&= \frac{1}{r(r-1)}\left(\begin{array}{l} (z(z-1) + (r-z)(r+z-1))(\lfloor n/r\rfloor)^2\\ +(2z(r-1))\lfloor n/r\rfloor + z(z-1)\end{array}\right)\\
&= \frac{1}{r(r-1)}\left(\begin{array}{l} (z^2 - z + r^2 - z^2 - r + z)(n/r - z/r)^2\\ +(2z(r-1))(n/r - z/r) + z(z-1)\end{array}\right)\\
&= \frac{1}{r^3(r-1)}\left((r^2-r)(n-z)^2 + r(r-1)2z(n-z) + r^2 z(z-1)\right)\\
&= \frac{1}{r^3(r-1)}\left(r(r-1)\left((n-z)^2 + 2z(n-z)\right) + r^2 z(z-1)\right)\\
&= \frac{1}{r^2(r-1)}\left((r-1)\left((n-z)^2 + 2z(n-z) + z^2\right) - z^2(r-1) + rz(z-1)\right)\\
&= \frac{1}{r^2(r-1)}\left((r-1)(n-z+z)^2 - rz + z^2\right)\\
&= \frac{n^2(r-1) - z(r-z)}{r^2(r-1)}. && (63)
\end{aligned}
$$

Using these expressions, we can compute the following expectations under complete randomization, assuming distinct treatments $s$ and $t$ and distinct units $i$ and $j$:

$$\begin{aligned}
\mathbb{E}(T_{is}T_{js}) &= = \mathbb{E}\left(\mathbb{E}(T_{is}T_{js}|\#T_s)\right) = \mathbb{E}\left(\frac{\#T_s(\#T_s-1)}{n(n-1)}\right) \\
&= \frac{\mathbb{E}[(\#T_s)^2] - \mathbb{E}[\#T_s]}{n(n-1)} = \frac{(n/r)^2 + z/r(1-z/r) - n/r}{n(n-1)} \\
&= \frac{(n/r)^2 - (z/r)^2 - (n/r - z/r)}{n(n-1)} \\
&= \frac{n^2 - z^2 - (nr - zr)}{n(n-1)r^2} \\
&= \frac{(n-z)(n+z) - (nr - zr)}{n(n-1)r^2} \\
&= \frac{(n-z)(n-r+z)}{n(n-1r^2)} = \frac{n(n-r) + z(r-z)}{n(n-1)r^2}, \qquad (64) \\
\mathbb{E}(T_{is}T_{jt}) &= \mathbb{E}[\mathbb{E}(T_{is}T_{jt}|\#T_s, \#T_t)] \\
&= \mathbb{E}\left(\frac{\#T_s\#T_t}{n(n-1)}\right) = \frac{\mathbb{E}(\#T_s\#T_t)}{n(n-1)} \\
&= \frac{n^2(r-1) - z(r-z)}{n(n-1)r^2(r-1)}. \qquad (65)
\end{aligned}$$

Under complete randomization, the expectation of the Horvitz-Thompson estimator is:

$$\mathbb{E}(\hat{\mu}_{s,\text{HT}}) = \mathbb{E}\left(\sum_{i=1}^{n} \frac{y_{is}T_{is}}{n/r}\right) = \sum_{i=1}^{n} \frac{y_{is}\mathbb{E}(T_{is})}{n/r} = \sum_{i=1}^{n} \frac{y_{is}(1/r)}{n/r} = \sum_{i=1}^{n} \frac{y_{is}}{n} = \mu_s. \qquad (66)$$

The variance of this estimator is derived as follows. By (64):

$$\begin{aligned}
\mathbb{E}(\hat{\mu}_{s,\mathrm{HT}}^2) &= \mathbb{E}\left(\left(\sum_{i=1}^{n} \frac{y_{is}T_{is}}{n/r}\right)^2\right) \\
&= \left(\frac{r^2}{n^2}\right)\left(\mathbb{E}\left(\sum_{i=1}^{n} y_{is}^2 T_{is}^2\right) + \mathbb{E}\left(\sum_{i=1}^{n}\sum_{j\neq i} y_{is}y_{js}T_{is}T_{js}\right)\right) \\
&= \left(\frac{r^2}{n^2}\right)\left(\mathbb{E}\left(\sum_{i=1}^{n} y_{is}^2 T_{is}\right) + \mathbb{E}\left(\sum_{i=1}^{n}\sum_{j\neq i} y_{is}y_{js}T_{is}T_{js}\right)\right) \\
&= \left(\frac{r^2}{n^2}\right)\left(\sum_{i=1}^{n} y_{is}^2 \mathbb{E}(T_{is}) + \sum_{i=1}^{n}\sum_{j\neq i} y_{is}y_{js}\mathbb{E}(T_{is}T_{js})\right) \\
&= \left(\frac{r^2}{n^2}\right)\left(\sum_{i=1}^{n} \frac{y_{is}^2}{r} + \sum_{i=1}^{n}\sum_{j\neq i} y_{is}y_{js}\frac{n(n-r)+z(r-z)}{r^2 n(n-1)}\right) \\
&= \left(\frac{r^2}{n^2}\right)\left[\sum_{i=1}^{n} \frac{y_{is}^2}{r} + \left(\frac{n(n-r)+z(r-z)}{r^2 n(n-1)}\right)\left(\left(\sum_{i=1}^{n} y_{is}\right)^2 - \sum_{i=1}^{n} y_{is}^2\right)\right] \\
&= \left(\frac{r}{n^2} - \frac{n(n-r)+z(r-z)}{n^3(n-1)}\right)\sum_{i=1}^{n} y_{is}^2 \\
&\quad + \left(\frac{n(n-r)+z(r-z)}{n^3(n-1)}\right)\left(\sum_{i=1}^{n} y_{is}\right)^2 \\
&= \left(\frac{rn(n-1)-n(n-r)-z(r-z)}{n^2(n-1)}\right)\sum_{i=1}^{n} \frac{y_{is}^2}{n} \\
&\quad + \left(\frac{n(n-r)+z(r-z)}{n(n-1)}\right)\left(\sum_{i=1}^{n} \frac{y_{is}}{n}\right)^2 \\
&= \left(\frac{n^2 r - n^2 - z(r-z)}{n^2(n-1)}\right)\sum_{i=1}^{n} \frac{y_{is}^2}{n} + \left(\frac{n(n-r)+z(r-z)}{n(n-1)}\right)\left(\sum_{i=1}^{n} \frac{y_{is}}{n}\right)^2 \\
&= \left(\frac{n^2(r-1) - z(r-z)}{n^2(n-1)}\right)\sum_{i=1}^{n} \frac{y_{is}^2}{n} + \left(\frac{n(n-r)+z(r-z)}{n(n-1)}\right)\left(\sum_{i=1}^{n} \frac{y_{is}}{n}\right)^2 .
\end{aligned} \tag{67}$$

It follows by (67) and (7) that:

$$
\begin{aligned}
& \mathrm{Var}(\hat{\mu}_{s,\mathrm{HT}}) = \mathbb{E}(\hat{\mu}_{s,\mathrm{HT}}^2) - (\mathbb{E}(\hat{\mu}_{s,\mathrm{HT}}))^2 \\
= \ & \left(\frac{n^2(r-1) - z(r-z)}{n^2(n-1)}\right) \sum_{i=1}^{n} \frac{y_{is}^2}{n} \\
& + \left(\frac{n(n-r) + z(r-z)}{n(n-1)}\right) \left(\sum_{i=1}^{n} \frac{y_{is}}{n}\right)^2 - \left(\sum_{i=1}^{n} \frac{y_{is}}{n}\right)^2 \\
= \ & \left(\frac{n^2(r-1) - z(r-z)}{n^2(n-1)}\right) \sum_{i=1}^{n} \frac{y_{is}^2}{n} \\
& + \left(\frac{n(n-r) + z(r-z) - n(n-1)}{n(n-1)}\right) \left(\sum_{i=1}^{n} \frac{y_{is}}{n}\right)^2 \\
= \ & \left(\frac{n^2(r-1) - z(r-z)}{n^2(n-1)}\right) \sum_{i=1}^{n} \frac{y_{is}^2}{n} + \left(\frac{n(1-r) + z(r-z)}{n(n-1)}\right) \left(\sum_{i=1}^{n} \frac{y_{is}}{n}\right)^2 \\
= \ & \left(\frac{n^2(r-1) - z(r-z)}{n^2(n-1)}\right) \sum_{i=1}^{n} \frac{y_{is}^2}{n} - \left(\frac{n(r-1) - z(r-z)}{n(n-1)}\right) \left(\sum_{i=1}^{n} \frac{y_{is}}{n}\right)^2 \\
= \ & \frac{r-1}{n-1} \left(\sum_{i=1}^{n} \frac{y_{is}^2}{n} - \left(\sum_{i=1}^{n} \frac{y_{is}}{n}\right)^2\right) - \frac{z(r-z)}{n^3(n-1)} \left(\sum_{i=1}^{n} y_{is}^2 - \left(\sum_{i=1}^{n} y_{is}\right)^2\right) \\
= \ & \frac{r-1}{n-1}\sigma_s^2 - \frac{z(r-z)}{n^3(n-1)} \left(-\sum_{i=1}^{n} \sum_{j \neq i} y_{is} y_{js}\right) \\
= \ & \frac{r-1}{n-1}\sigma_s^2 + \frac{z(r-z)}{n^3(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} y_{is} y_{js}.
\end{aligned} \tag{68}
$$

Now we derive the covariance. Note that:

$$\begin{aligned}
& \mathbb{E}\left(\sum_{i=1}^{n}\frac{y_{is}T_{is}}{n/r}\sum_{i=1}^{n}\frac{y_{it}T_{it}}{n/r}\right) \\
= & \mathbb{E}\left(\sum_{i=1}^{n}\sum_{j\neq i}\frac{y_{is}y_{jt}T_{is}T_{jt}}{(n/r)^2}\right)+\mathbb{E}\left(\sum_{i=1}^{n}\frac{y_{is}y_{it}T_{is}T_{it}}{(n/r)^2}\right) \\
= & \sum_{i=1}^{n}\sum_{j\neq i}\frac{y_{is}y_{jt}}{(n/r)^2}\mathbb{E}\left(T_{is}T_{jt}\right)+\sum_{i=1}^{n}\frac{y_{is}y_{it}}{(n/r)^2}\mathbb{E}\left(T_{is}T_{it}\right) \\
= & \sum_{i=1}^{n}\sum_{j\neq i}\frac{y_{is}y_{jt}}{(n/r)^2}\frac{n^2(r-1)-z(r-z)}{n(n-1)r^2(r-1)}+0 \\
= & \sum_{i=1}^{n}\sum_{j\neq i}\frac{y_{is}y_{jt}(n^2(r-1)-z(r-z))}{n^3(n-1)(r-1)} \\
= & \sum_{i=1}^{n}\sum_{j\neq i}\frac{y_{is}y_{jt}}{n(n-1)}-\sum_{i=1}^{n}\sum_{j\neq i}\frac{y_{is}y_{jt}z(r-z)}{n^3(n-1)(r-1)}. \qquad (69)
\end{aligned}$$

Thus, using $\text{cov}(X,Y)=\mathbb{E}(XY)-\mathbb{E}(X)\mathbb{E}(Y)$ and applying (69) and (8), we have:

$$\begin{aligned}
\text{cov}(\hat{\mu}_{s,\text{HT}},\hat{\mu}_{t,\text{HT}}) &= \mathbb{E}(\hat{\mu}_{s,\text{HT}}\hat{\mu}_{t,\text{HT}})-\mathbb{E}(\hat{\mu}_{s,\text{HT}})\mathbb{E}(\hat{\mu}_{t,\text{HT}}) \\
&= \mathbb{E}\left(\sum_{i=1}^{n}\frac{y_{is}T_{is}}{n/r}\sum_{i=1}^{n}\frac{y_{it}T_{it}}{n/r}\right)-\sum_{i=1}^{n}\frac{y_{is}}{n}\sum_{i=1}^{n}\frac{y_{it}}{n} \\
&= \sum_{i=1}^{n}\sum_{j\neq i}\frac{y_{is}y_{jt}}{n(n-1)}-\sum_{i=1}^{n}\sum_{j\neq i}\frac{y_{is}y_{jt}z(r-z)}{n^3(n-1)(r-1)}-\sum_{i=1}^{n}\frac{y_{is}}{n}\sum_{i=1}^{n}\frac{y_{it}}{n} \\
&= \frac{-\gamma_{st}}{n-1}-\sum_{i=1}^{n}\sum_{j\neq i}\frac{y_{is}y_{jt}z(r-z)}{n^3(n-1)(r-1)}. \qquad (70)
\end{aligned}$$

This proves the two lemmas.

# B Proof of Lemmas 6 and 7

To help the reader, we suppress the block index in the following derivations, identifying units by a single index.

Note that, under complete randomization and for distinct treatments $s$ and $t$ and distinct units $i$ and $j$, the following expectations hold:

$$
\begin{aligned}
\mathbb{E}\left(\frac{T_{is}}{\#T_s - 1}\right) &= \mathbb{E}\left[\mathbb{E}\left(\frac{T_{is}}{\#T_s - 1}\middle| \#T_s\right)\right] \\
&= \mathbb{E}\left(\frac{\frac{\#T_s}{n}}{\#T_s - 1}\right) = \frac{1}{n}\mathbb{E}\left(\frac{\#T_s}{\#T_s - 1}\right) \\
&= \frac{1}{n}\mathbb{E}\left(\frac{\#T_s - 1}{\#T_s - 1}\right) + \frac{1}{n}\mathbb{E}\left(\frac{1}{\#T_s - 1}\right) \\
&= \frac{1}{n} + \frac{1}{n}\mathbb{E}\left(\frac{1}{\#T_s - 1}\right), \qquad (71) \\
\mathbb{E}\left(\frac{T_{is}}{\#T_s(\#T_s - 1)}\right) &= \mathbb{E}\left[\mathbb{E}\left(\frac{T_{is}}{\#T_s(\#T_s - 1)}\middle| \#T_s\right)\right] \\
&= \mathbb{E}\left(\frac{\frac{\#T_s}{n}}{\#T_s(\#T_s - 1)}\right) = \frac{1}{n}\mathbb{E}\left(\frac{\#T_s}{\#T_s(\#T_s - 1)}\right) \\
&= \frac{1}{n}\mathbb{E}\left(\frac{1}{\#T_s - 1}\right), \qquad (72) \\
\mathbb{E}\left(\frac{T_{is}T_{js}}{\#T_s(\#T_s - 1)}\right) &= \mathbb{E}\left[\mathbb{E}\left(\frac{T_{is}T_{js}}{\#T_s(\#T_s - 1)}\middle| \#T_s\right)\right] \\
&= \mathbb{E}\left(\frac{\frac{\#T_s}{n}\frac{\#T_s - 1}{n-1}}{\#T_s(\#T_s - 1)}\right) \\
&= \frac{1}{n(n-1)}\mathbb{E}\left(\frac{\#T_s(\#T_s - 1)}{\#T_s(\#T_s - 1)}\right) = \frac{1}{n(n-1)}. \qquad (73)
\end{aligned}
$$

We show that $\mathbb{E}(\hat{\sigma}^2_{s,\text{samp}}) = \sigma^2_s$. The fact that $\mathbb{E}\left[\widehat{\text{Var}}(\hat{\mu}_{s,\text{samp}})\right] = \text{Var}(\hat{\mu}_{s,\text{samp}})$ follows immediately.

First, note that:

$$\begin{aligned}
&\sum_{i=1}^{n} T_{is}\left(y_{is}T_{is} - \sum_{i=1}^{n}\frac{y_{is}T_{is}}{\#T_s}\right)^2 \\
= &\sum_{i=1}^{n} T_{is}(y_{is}T_{is})^2 - 2\sum_{i=1}^{n} T_{is}\left(y_{is}T_{is}\sum_{i=1}^{n}\frac{y_{is}T_{is}}{\#T_s}\right) + \sum_{i=1}^{n} T_{is}\left(\sum_{i=1}^{n}\frac{y_{is}T_{is}}{\#T_s}\right)^2 \\
= &\sum_{i=1}^{n} y_{is}^2 T_{is} - 2\sum_{i=1}^{n}\left(y_{is}T_{is}\sum_{i=1}^{n}\frac{y_{is}T_{is}}{\#T_s}\right) + \sum_{i=1}^{n} T_{is}\left(\sum_{i=1}^{n}\frac{y_{is}T_{is}}{\#T_s}\right)^2 \\
= &\sum_{i=1}^{n} y_{is}^2 T_{is} - 2\#T_s\left(\sum_{i=1}^{n}\frac{y_{is}T_{is}}{\#T_s}\right)^2 + \#T_s\left(\sum_{i=1}^{n}\frac{y_{is}T_{is}}{\#T_s}\right)^2 \\
= &\sum_{i=1}^{n} y_{is}^2 T_{is} - \#T_s\left(\sum_{i=1}^{n}\frac{y_{is}T_{is}}{\#T_s}\right)^2 . \qquad (74)
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbb{E}(\hat{\sigma}^2_{s,\text{diff}}) &= \mathbb{E}\left(\frac{n-1}{n}\sum_{i=1}^{n}\frac{T_{is}\left(y_{is} - \sum_{i=1}^{n}\frac{y_{is}T_{is}}{\#T_s}\right)^2}{\#T_s - 1}\right) \\
&= \frac{n-1}{n}\mathbb{E}\left(\frac{\sum_{i=1}^{n} y_{is}^2 T_{is} - \#T_s\left(\sum_{i=1}^{n}\frac{y_{is}T_{is}}{\#T_s}\right)^2}{\#T_s - 1}\right) \\
&= \frac{n-1}{n}\left(\sum_{i=1}^{n} y_{is}^2\mathbb{E}\left(\frac{T_{is}}{\#T_s - 1}\right) - \mathbb{E}\left(\frac{(\sum_{i=1}^{n} y_{is}T_{is})^2}{\#T_{is}(\#T_{is} - 1)}\right)\right). \qquad (75)
\end{aligned}$$

By (71):

$$\sum_{i=1}^{n} y_{is}^2\mathbb{E}\left(\frac{T_{is}}{\#T_s - 1}\right) = \frac{1}{n}\sum_{i=1}^{n} y_{is}^2 + \frac{1}{n}\mathbb{E}\left(\frac{1}{\#T_s - 1}\right)\sum_{i=1}^{n} y_{is}^2. \qquad (76)$$

By (72) and (73):

$$
\begin{aligned}
& \mathbb{E}\left(\frac{\left(\sum_{i=1}^{n} y_{is}T_{is}\right)^2}{\#T_{is}(\#T_{is}-1)}\right) \\
= \;& \mathbb{E}\left(\frac{\sum_{i=1}^{n}(y_{is}T_{is})^2}{\#T_{is}(\#T_{is}-1)}\right) + \mathbb{E}\left(\frac{\sum_{i=1}^{n}\sum_{j\neq i} y_{is}y_{js}T_{is}T_{js}}{\#T_{is}(\#T_{is}-1)}\right) \\
= \;& \sum_{i=1}^{n} y_{is}^2 \mathbb{E}\left(\frac{T_{is}}{\#T_{is}(\#T_{is}-1)}\right) + \sum_{j\neq i} y_{is}y_{js}\mathbb{E}\left(\frac{T_{is}T_{js}}{\#T_{is}(\#T_{is}-1)}\right) \\
= \;& \frac{1}{n}\mathbb{E}\left(\frac{1}{\#T_s-1}\right)\sum_{i=1}^{n} y_{is}^2 + \frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j\neq i} y_{is}y_{js} \\
= \;& \frac{1}{n}\mathbb{E}\left(\frac{1}{\#T_s-1}\right)\sum_{i=1}^{n} y_{is}^2 + \frac{1}{n(n-1)}\left(\sum_{i=1}^{n} y_{is}\right)^2 - \frac{1}{n(n-1)}\sum_{i=1}^{n} y_{is}^2.
\end{aligned}
\tag{77}
$$

Thus, by (76), (77), and (7), it follows that:

$$
\begin{aligned}
\mathbb{E}(\hat{\sigma}_{s,\text{diff}}^2) &= \frac{n-1}{n}\left(\sum_{i=1}^{n} y_{is}^2\mathbb{E}\left(\frac{T_{is}}{\#T_s-1}\right) - \mathbb{E}\left(\frac{\left(\sum_{i=1}^{n} y_{is}T_{is}\right)^2}{\#T_{is}(\#T_{is}-1)}\right)\right) \\
&= \frac{n-1}{n}\left(\frac{1}{n}\sum_{i=1}^{n} y_{is}^2 + \frac{1}{n}\mathbb{E}\left(\frac{1}{\#T_s-1}\right)\sum_{i=1}^{n} y_{is}^2\right) \\
&\quad -\frac{n-1}{n}\left(\frac{1}{n}\mathbb{E}\left(\frac{1}{\#T_s-1}\right)\sum_{i=1}^{n} y_{is}^2 - \frac{1}{n(n-1)}\left(\sum_{i=1}^{n} y_{is}\right)^2 + \frac{1}{n(n-1)}\sum_{i=1}^{n} y_{is}^2\right) \\
&= \frac{n-1}{n}\left(\frac{1}{n-1}\sum_{i=1}^{n} y_{is}^2 - \frac{1}{n(n-1)}\left(\sum_{i=1}^{n} y_{is}\right)^2\right) \\
&= \frac{n-1}{n}\left(\frac{n}{n-1}\sum_{i=1}^{n}\frac{y_{is}^2}{n} - \frac{n}{n-1}\left(\sum_{i=1}^{n}\frac{y_{is}}{n}\right)^2\right) \\
&= \frac{n-1}{n}\left(\frac{n}{n-1}\sigma_s^2\right) = \sigma_s^2.
\end{aligned}
\tag{78}
$$

We now focus on estimating the variance of Horvitz-Thompson estimators. By (64), for any treatment $s$, the following expectation holds under complete randomization.

$$\begin{aligned}
& \mathbb{E}\left(\frac{n(n-1)r^2}{n(n-r)+z(r-z)}\sum_{i=1}^{n}\sum_{j\neq i} y_{is}y_{js}T_{is}T_{js}\right) \\
= & \frac{n(n-1)r^2}{n(n-r)+z(r-z)}\sum_{i=1}^{n}\sum_{j\neq i} y_{is}y_{js}\mathbb{E}(T_{is}T_{js}) \\
= & \frac{n(n-1)r^2}{n(n-r)+z(r-z)}\sum_{i=1}^{n}\sum_{j\neq i} y_{is}y_{js}\frac{n(n-r)+z(r-z)}{n(n-1)r^2} \\
= & \sum_{i=1}^{n}\sum_{j\neq i} y_{is}y_{js},
\end{aligned} \tag{79}$$

Applying (79), we show unbiasedness of the Horvitz-Thompson variance estimator under complete randomization:

$$\begin{aligned}
\mathbb{E}\left(\hat{\sigma}^2_{s,\mathrm{HT}}\right) &= \mathbb{E}\left(\frac{(n-1)r}{n^2}\sum_{i=1}^{n} y_{is}^2 T_{is}\right) \\
&\quad -\mathbb{E}\left(\frac{(n-1)r^2}{n^2(n-r)+nz(r-z)}\sum_{i=1}^{n}\sum_{j\neq i} y_{is}y_{js}T_{is}T_{js}\right) \\
&= \frac{(n-1)r}{n^2}\sum_{i=1}^{n} y_{is}^2\mathbb{E}(T_{is}) \\
&\quad -\frac{1}{n^2}\mathbb{E}\left(\frac{n(n-1)r^2}{n(n-r)+z(r-z)}\sum_{i=1}^{n}\sum_{j\neq i} y_{is}y_{js}T_{is}T_{js}\right) \\
&= \frac{(n-1)r}{n^2}\sum_{i=1}^{n} y_{is}^2(1/r) - \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j\neq i} y_{is}y_{js} \\
&= \left(\frac{1}{n}-\frac{1}{n^2}\right)\sum_{i=1}^{n} y_{is}^2 - \frac{1}{n^2}\left(\left(\sum_{i=1}^{n} y_{is}\right)^2 - \sum_{i=1}^{n} y_{is}^2\right) \\
&= \frac{1}{n}\sum_{i=1}^{n} y_{is}^2 - \frac{1}{n^2}\left(\sum_{i=1}^{n} y_{is}\right)^2 = \sum_{i=1}^{n}\frac{y_{is}^2}{n} - \left(\sum_{i=1}^{n}\frac{y_{is}}{n}\right)^2 = \sigma_s^2.
\end{aligned} \tag{80}$$

Thus, by (64) and (80), we show unbiasedness of the variance estimator $\widehat{\text{Var}}(\hat{\mu}_{s,\text{HT}})$ under complete randomization:

$$\begin{aligned}
\mathbb{E}(\widehat{\text{Var}}(\hat{\mu}_{s,\text{HT}})) &= \mathbb{E}\left(\frac{r-1}{n-1}\hat{\sigma}^2_{s,\text{HT}} + \frac{r^2 z(r-z)}{n^3(n-r)+n^2 z(r-z)}\sum_{i=1}^{n}\sum_{j\neq i} y_{is}y_{js}T_{is}T_{js}\right) \\
&= \frac{r-1}{n-1}\mathbb{E}\left(\hat{\sigma}^2_{s,\text{HT}}\right) + \mathbb{E}\left(\frac{r^2 z(r-z)}{n^3(n-r)+n^2 z(r-z)}\sum_{i=1}^{n}\sum_{j\neq i} y_{is}y_{js}T_{is}T_{js}\right) \\
&= \frac{r-1}{n-1}\sigma^2_s + \mathbb{E}\left(\frac{z(r-z)}{n^3(n-1)}\frac{n(n-1)r^2}{n(n-r)+z(r-z)}\sum_{i=1}^{n}\sum_{j\neq i} y_{is}y_{js}T_{is}T_{js}\right) \\
&= \frac{r-1}{n-1}\sigma^2_s + \frac{z(r-z)}{n^3(n-1)}\mathbb{E}\left(\frac{n(n-1)r^2}{n(n-r)+z(r-z)}\sum_{i=1}^{n}\sum_{j\neq i} y_{is}y_{js}T_{is}T_{js}\right) \\
&= \frac{r-1}{n-1}\sigma^2_s + \frac{z(r-z)}{n^3(n-1)}\sum_{i=1}^{n}\sum_{j\neq i} y_{is}y_{js} = \text{Var}(\hat{\mu}_{s,\text{HT}}). \qquad (81)
\end{aligned}$$

From (65), it follows that:

$$\begin{aligned}
&\mathbb{E}\left(\frac{n(n-1)r^2(r-1)}{n^2(r-1)-z(r-z)}\sum_{i=1}^{n}\sum_{j\neq i} y_{is}y_{jt}T_{is}T_{jt}\right) \\
&= \frac{n(n-1)r^2(r-1)}{n^2(r-1)-z(r-z)}\sum_{i=1}^{n}\sum_{j\neq i} y_{is}y_{jt}\mathbb{E}(T_{is}T_{jt}) \\
&= \frac{n(n-1)r^2(r-1)}{n^2(r-1)-z(r-z)}\sum_{i=1}^{n}\sum_{j\neq i} y_{is}y_{jt}\frac{n^2(r-1)-z(r-z)}{n(n-1)r^2(r-1)} \\
&= \sum_{i=1}^{n}\sum_{j\neq i} y_{is}y_{jt} \qquad (82)
\end{aligned}$$

Thus, by (82):

$$\begin{aligned}
& \mathbb{E}\left(\frac{2r^2 z(r-z)}{n^4(r-1) - n^2 z(r-z)} \sum_{i=1}^{n} \sum_{j\neq i} y_{is} y_{jt} T_{is} T_{jt}\right) \\
= & \mathbb{E}\left(\frac{2z(r-z)}{n^2} \frac{r^2}{n^2(r-1) - z(r-z)} \sum_{i=1}^{n} \sum_{j\neq i} y_{is} y_{jt} T_{is} T_{jt}\right) \\
= & \mathbb{E}\left(\frac{2z(r-z)}{n^3(n-1)(r-1)} \frac{n(n-1)r^2(r-1)}{n^2(r-1) - z(r-z)} \sum_{i=1}^{n} \sum_{j\neq i} y_{is} y_{jt} T_{is} T_{jt}\right) \\
= & \frac{2z(r-z)}{n^3(n-1)(r-1)} \mathbb{E}\left(\frac{n(n-1)r^2(r-1)}{n^2(r-1) - z(r-z)} \sum_{i=1}^{n} \sum_{j\neq i} y_{is} y_{jt} T_{is} T_{jt}\right) \\
= & \frac{2z(r-z)}{n^3(n-1)(r-1)} \sum_{i=1}^{n} \sum_{j\neq i} y_{is} y_{jt}. \qquad (83)
\end{aligned}$$

# C Proof of Theorem 9

Consider a block of size $n_c$. Let $\mathbf{S}_{n_c}$ denote an arbitrary subset of $\{1, \ldots, n\}$ of size $|\mathbf{S}_{n_c}| = n_c$. Let $\mathbf{1}(i \in S_{n_c})$ denote an indicator function: $\mathbf{1}(i \in \mathbf{S}_{n_c}) = 1$ if $i \in \mathbf{S}_{n_c}$; otherwise, $\mathbf{1}(i \in \mathbf{S}_{n_c}) = 0$. Possible values of the within-block variance $\sigma^2_{cs}$ are

$$\frac{1}{n_c} \sum_{i \in \mathbf{S}_{n_c}} y_{is}^2 \mathbf{1}(i \in \mathbf{S}_{n_c}) - \frac{1}{n_c^2} \left(\sum_{i \in \mathbf{S}_{n_c}} y_{is} \mathbf{1}(i \in \mathbf{S}_{n_c})\right)^2 \qquad (84)$$

and possible values of the variance $\sigma^2_{c(s+t)}$ are

$$\begin{aligned}
& \frac{1}{n_c} \sum_{i \in \mathbf{S}_{n_c}} y_{is}^2 \mathbf{1}(i \in \mathbf{S}_{n_c}) + y_{it}^2 \mathbf{1}(i \in \mathbf{S}_{n_c}) \\
& - \frac{1}{n_c^2} \left(\sum_{i \in \mathbf{S}_{n_c}} y_{is} \mathbf{1}(i \in \mathbf{S}_{n_c}) + y_{it} \mathbf{1}(i \in \mathbf{S}_{n_c})\right)^2. \qquad (85)
\end{aligned}$$

Under completely randomized blocking, the probability that block $c$ is comprised of the units in $\mathbf{S}_{n_c}$ is $\binom{n}{n_c}$. Thus, the expectation of $\sigma_{cs}$ is

$$
\begin{aligned}
\mathbb{E}(\sigma_{cs}^2) &= \binom{n}{n_c}^{-1} \sum_{\mathbf{S}_{n_c}} \left( \frac{1}{n_c} \sum_{i \in \mathbf{S}_{n_c}} y_{is}^2 \mathbf{1}(i \in \mathbf{S}_{n_c}) - \frac{1}{n_c^2} \left( \sum_{i \in \mathbf{S}_{n_c}} y_{is} \mathbf{1}(i \in \mathbf{S}_{n_c}) \right)^2 \right) \\
&= \binom{n}{n_c}^{-1} \sum_{\mathbf{S}_{n_c}} \left( \frac{1}{n_c} \sum_{i \in \mathbf{S}_{n_c}} y_{is}^2 \mathbf{1}(i \in \mathbf{S}_{n_c}) - \frac{1}{n_c^2} \sum_{i \in \mathbf{S}_{n_c}} y_{is}^2 \mathbf{1}(i \in \mathbf{S}_{n_c}) \right) \\
&\quad - \binom{n}{n_c}^{-1} \sum_{\mathbf{S}_{n_c}} \frac{1}{n_c^2} \sum_{i \neq j \in \mathbf{S}_{n_c}} y_{is} y_{js} \mathbf{1}(i \in \mathbf{S}_{n_c}) \mathbf{1}(j \in \mathbf{S}_{n_c}) \\
&= \binom{n}{n_c}^{-1} \frac{n_c - 1}{n_c^2} \sum_{\mathbf{S}_{n_c}} \sum_{i \in \mathbf{S}_{n_c}} y_{is}^2 \mathbf{1}(i \in \mathbf{S}_{n_c}) \\
&\quad - \binom{n}{n_c}^{-1} \frac{1}{n_c^2} \sum_{\mathbf{S}_{n_c}} \sum_{i \neq j \in \mathbf{S}_{n_c}} y_{is} y_{js} \mathbf{1}(i \in \mathbf{S}_{n_c}) \mathbf{1}(j \in \mathbf{S}_{n_c}) \\
&= \binom{n}{n_c}^{-1} \frac{n_c - 1}{n_c^2} \sum_{i=1}^{n} \sum_{\mathbf{S}_{n_c}} y_{is}^2 \mathbf{1}(i \in \mathbf{S}_{n_c}) \\
&\quad - \binom{n}{n_c}^{-1} \frac{1}{n_c^2} \sum_{i=1}^{n} \sum_{j \neq i} \sum_{\mathbf{S}_{n_c}} y_{is} y_{js} \mathbf{1}(i \in \mathbf{S}_{n_c}) \mathbf{1}(j \in \mathbf{S}_{n_c}) \\
&= \binom{n}{n_c}^{-1} \frac{n_c - 1}{n_c^2} \sum_{i=1}^{n} y_{is}^2 \binom{n-1}{n_c - 1} - \binom{n}{n_c}^{-1} \frac{1}{n_c^2} \sum_{i=1}^{n} \sum_{j \neq i} y_{is} y_{js} \binom{n-2}{n_c - 2}.
\end{aligned}
\tag{86}
$$

Now since

$$
\binom{n-1}{n_c - 1} \binom{n}{n_c}^{-1} = \frac{(n-1)! n_c! (n - n_c)!}{(n_c - 1)! (n - n_c)! n!} = \frac{n_c}{n} \tag{87}
$$

$$
\binom{n-2}{n_c - 2} \binom{n}{n_c}^{-1} = \frac{(n-2)! n_c! (n - n_c)!}{(n_c - 2)! (n - n_c)! n!} = \frac{n_c (n_c - 1)}{n(n-1)}. \tag{88}
$$

It follows that

$$
\begin{aligned}
\sigma_{cs}^2 &= \binom{n}{n_c}^{-1} \frac{n_c - 1}{n_c^2} \sum_{i=1}^{n} y_{is}^2 \binom{n-1}{n_c - 1} - \binom{n}{n_c}^{-1} \frac{1}{n_c^2} \sum_{i=1}^{n} \sum_{j \neq i} y_{is} y_{js} \binom{n-2}{n_c - 2} \\
&= \frac{n_c}{n} \frac{n_c - 1}{n_c^2} \sum_{i=1}^{n} y_{is}^2 - \frac{n_c(n_c - 1)}{n(n-1)} \frac{1}{n_c^2} \sum_{i=1}^{n} \sum_{j \neq i} y_{is} y_{js} \\
&= \frac{n_c - 1}{n(n_c)} \sum_{i=1}^{n} y_{is}^2 - \frac{n_c - 1}{n(n-1)n_c} \sum_{i=1}^{n} \sum_{j \neq i} y_{is} y_{js} \\
&= \frac{n_c - 1}{n(n_c)} \sum_{i=1}^{n} y_{is}^2 - \frac{n(n_c - 1)}{(n-1)n_c} \sum_{i=1}^{n} \sum_{j \neq i} \frac{y_{is} y_{js}}{n^2} \\
&= \frac{n_c - 1}{n(n_c)} \sum_{i=1}^{n} y_{is}^2 - \frac{n(n_c - 1)}{(n-1)n_c} \left( \sum_{i=1}^{n} \frac{y_{is}}{n} \right)^2 + \frac{n_c - 1}{n(n-1)n_c} \sum_{i=1}^{n} y_{is}^2 \\
&= \frac{n_c - 1}{n(n_c)} \sum_{i=1}^{n} y_{is}^2 - \frac{n(n_c - 1)}{(n-1)n_c} \left( \sum_{i=1}^{n} \frac{y_{is}}{n} \right)^2 + \frac{n_c - 1}{n(n-1)n_c} \sum_{i=1}^{n} y_{is}^2 \\
&= \frac{(n-1)(n_c - 1)}{n(n-1)n_c} \sum_{i=1}^{n} y_{is}^2 - \frac{n(n_c - 1)}{(n-1)n_c} \left( \sum_{i=1}^{n} \frac{y_{is}}{n} \right)^2 + \frac{n_c - 1}{n(n-1)n_c} \sum_{i=1}^{n} y_{is}^2 \\
&= \frac{n(n_c - 1)}{n(n-1)n_c} \sum_{i=1}^{n} y_{is}^2 - \frac{n(n_c - 1)}{(n-1)n_c} \left( \sum_{i=1}^{n} \frac{y_{is}}{n} \right)^2 \\
&= \frac{n(n_c - 1)}{(n-1)n_c} \sum_{i=1}^{n} \frac{y_{is}^2}{n} - \frac{n_c(n_c - 1)}{n(n-1)} \left( \sum_{i=1}^{n} y_{is} \right)^2 = \frac{n(n_c - 1)}{(n-1)n_c} \sigma_s^2. \qquad (89)
\end{aligned}
$$

Likewise, substituting $y_{is} + y_{it}$ in for $y_{is}$, we obtain:

$$\sigma_{c(s+t)}^2 = \frac{n(n_c - 1)}{(n-1)n_c} \sigma_{s+t}^2. \qquad (90)$$

Therefore

$$
\begin{aligned}
& \mathbb{E}\left(\frac{n_c^2}{(n_c-1)n^2}[(r-2)(\sigma_{cs}^2+\sigma_{ct}^2)+\sigma_{c(s+t)}^2]\right) \\
= \;& \frac{n_c^2}{(n_c-1)n^2}[(r-2)(\mathbb{E}(\sigma_{cs}^2)+\mathbb{E}(\sigma_{ct}^2))+\mathbb{E}(\sigma_{c(s+t)}^2)] \\
= \;& \frac{n_c^2}{(n_c-1)n^2}\left[(r-2)\left(\frac{n(n_c-1)}{(n-1)n_c}\sigma_s^2+\frac{n(n_c-1)}{(n-1)n_c}\sigma_t^2\right)+\frac{n(n_c-1)}{(n-1)n_c}\sigma_{s+t}^2\right] \\
= \;& \frac{n_c^2}{(n_c-1)n^2}\frac{n(n_c-1)}{(n-1)n_c}\left[(r-2)(\sigma_s^2+\sigma_t^2)+\sigma_{s+t}^2\right] \\
= \;& \frac{n_c}{n(n-1)}\left[(r-2)(\sigma_s^2+\sigma_t^2)+\sigma_{s+t}^2\right]. \qquad (91)
\end{aligned}
$$

Finally, it follows that the expected difference in variances is:

$$
\begin{aligned}
& \mathbb{E}\left[\sum_{c=1}^{b}\frac{n_c^2}{(n-1)\sum n_c^2}[(r-2)(\sigma_s^2+\sigma_t^2)+\sigma_{s+t}^2]\right] \\
& -\mathbb{E}\left[\sum_{c=1}^{b}\frac{n_c^2}{(n_c-1)n^2}[(r-2)(\sigma_{cs}^2+\sigma_{ct}^2)+\sigma_{c(s+t)}^2]\right] \\
= \;& \sum_{c=1}^{b}\frac{n_c^2}{(n-1)\sum n_c^2}[(r-2)(\sigma_s^2+\sigma_t^2)+\sigma_{s+t}^2] \\
& -\sum_{c=1}^{b}\frac{n_c}{n(n-1)}\left[(r-2)(\sigma_s^2+\sigma_t^2)+\sigma_{s+t}^2\right] \\
= \;& \frac{(r-2)(\sigma_s^2+\sigma_t^2)+\sigma_{s+t}^2}{n-1}\left(\frac{\sum n_c^2}{\sum n_c^2}-\frac{\sum n_c}{n}\right) \\
= \;& \frac{(r-2)(\sigma_s^2+\sigma_t^2)+\sigma_{s+t}^2}{n-1}(0)=0. \qquad (92)
\end{aligned}
$$

That is, in expectation, the variance of estimates of the SATE under block randomization with random blocks be the same as those under complete randomization.

# References


Abadie, A. and Imbens, G. (2008). Estimation of the conditional variance in paired experiments. *Annales d'Economie et de Statistique*, No. 91-92:175–187.

Cochran, W. (1977). *Sampling techniques*. Wiley, New York, NY.

Fisher, R. A. (1926). The arrangement of field experiments. *Journal of the Ministry of Agriculture of Great Britain*, 33:503–513.

Hardy, G., Littlewood, J., and Pólya, G. (1952). *Inequalities*. Cambridge University Press, second edition.

Holland, P. W. (1986). Statistics and causal inference. *Journal of the American statistical Association*, 81(396):945–960.

Horvitz, D. G. and Thompson, D. J. (1952). A generalization of sampling without replacement from a finite universe. *Journal of the American Statistical Association*, 47(260):663–685.

Imai, K. (2008). Variance identification and efficiency analysis in randomized experiments under the matched-pair design. *Statistics in medicine*, 27(24):4857–4873.

Imbens, G. W. (2011). Experimental design for unit and cluster randomized trials. Working Paper.

Lohr, S. (1999). *Sampling: Design and Analysis*. Duxbury Press, Pacific Grove, CA.

Miratrix, L. W., Sekhon, J. S., and Yu, B. (2013). Adjusting treatment effect estimates by post-stratification in randomized experiments. *Journal of the Royal Statistical Society, Series B*, 75(2):369–396.

Neyman, J. (1935). Statistical problems in agricultural experimentation (with discussion). *Supplement of Journal of the Royal Statistical Society*, 2:107 180.

Rubin, D. B. (1974). Estimating causal effects of treatments in randomized and nonrandomized studies. *Journal of Educational Psychology; Journal of Educational Psychology*, 66(5):688.

Splawa-Neyman, J., Dabrowska, D., and Speed, T. (1990). On the application of probability theory to agricultural experiments. essay on principles. section 9. *Statistical Science*, 5(4):465–472.